Periodic, semi-clean and CJ elements

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6. semiclean $\exists p \in Per(R), \exists u \in U(R)$ s.t. $a = p + u$. $Scl(R)$
7. strongly clean if $a = e + u$ is clean and $ue = eu$.

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1. *for all $k \in \mathbb{N}$ and any $j \geq m$, $a^j = a^{j+k(l-m)}$.*
2. *$a^{m(l-m)}$ is an idempotent.*
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(1) We have $a^m = a^m a^{l-m} = a^m a^{2(l-m)} = \cdots = a^{m+k(l-m)}$ and hence also $a^j = a^{j+k(l-m)}$ for any $j \geq m$ and all $k \in \mathbb{N}$.

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(2) Using (1), we have
$(a^{m(l-m)})^2 = a^{m(l-m)+m(l-m)} = a^{m(l-m)}$. □

Theorem

Let $p = \sum_{i=0}^{n} p_i x^i \in R[x]$ *be such that*

1. $p^l = p^m$, *for some* $l > m$,
2. $[p_0, p_i] = 0$, *for every* $0 \leq i \leq n$,
3. $(l - m)p_i \neq 0$ *if* $p_i \neq 0$, *for every* $0 \leq i \leq n$.

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Corollary

If $m = 1$ *in the above theorem, under the same conditions we get that the potent polynomials* $p \in R[x]$ *belong to the base ring* R.

Remark

The polynomial $p(x) = 4x + 1 \in (\mathbb{Z}/8\mathbb{Z})[x]$ is such that $p(x)^3 = p(x)$. This shows that the condition on the coefficients cannot be omitted.

Proposition

Let $p(x) = \sum_{i=0}^{n} p_i x^i \in Per(R[x])$ *be such that* $p_i p_0 = p_0 p_i$ *for* $1 \leq i \leq n$. *Suppose there exists a natural number* q *such that* $qp_i = 0$ *for* $1 \leq i \leq n$. *Then* $p - p_0$ *is nilpotent.*

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Remark

The above results admit generalizations for $\mathbb{N}$-graded rings. $R = \oplus_{i \in \mathbb{N}} R_i$ where R_i are additive groups and the product of R is such that $R_i R_j \subseteq R_{i+j}$. In particular, we can get results on $Per(S)$ when $S = R[x_1, \ldots, x_n]$.

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Theorem

A ring R is periodic if and only if $R/J(R)$ is periodic and $J(R)$ is nil.

Matrices over periodic rings

Let us mention some important results related to matrices over periodic rings.

Theorem (A. Bouzidi, A. Cherchem, A. Leroy; 2020)

If R is a periodic ring then $M_n(R)$ is also periodic in the following cases:

1. *R is Artinian.*
2. *R is right (left) Noetherian and $J(R)$ is nilpotent.*
3. *R is P.I.*

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Corollary

If R is 2-primal and $\sum a_i x^i \in Per(R[X])$, then $a_0 \in Per(R)$ and $a_i \in Nil(R)$ for $i \geq 1$. Thus in this case we have $Per(R[x]) \subseteq Per(R) + Nil(R)[x]x$.

Example

Suppose $R = \mathbb{Z}[y]/(y^2)$. R is a commutative ring hence 2-primal. Consider $1 + yx \in R[x]$, 1 is perodic and y is nilpotent. But $(1 + yx)^n = 1 + nyx$ is not periodic for any $n \in \mathbb{N}$. This shows the converse inclusion of the above does not always hold.

Definition

An element $a \in R$ is semiclean if there exist a periodic element $p \in R$ and a unit $u \in U(R)$ such that $a = p + u$. The set $Scl(R)$ denotes the set of semiclean elements. The ring R is semiclean if $Scl(R) = R$.

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Proposition

1. $Scl(R) + J(R) \subseteq Scl(R)$.
2. $Scl(R[x]) \cap R = Scl(R)$.
3. *If R is a domain, then the semiclean elements are units or sum of two units.*

Among the following equivalent statements, 3 and 4 were given by Kanwar, Leroy, and Matczuk.

Proposition

Let R be a ring, then the following are equivalent:

1. R *is 2 primal.*
2. $R[x]$ *is 2 primal.*
3. $Cl(R[x]) = Cl(R) + Nil(R)[x]x$.
4. $U(R[x]) = U(R) + Nil(R)[x]x$.
5. $Scl(R[x]) = Scl(R) + Nil(R)[x]x$.

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We have the following easy relations between these rings.

$$CN \Rightarrow CU, \qquad CJ \Rightarrow CU$$

.

Examples

(1) Every commutative ring is CJ.
(2) Every homomorphic image of a CJ ring is CJ.
(3) $C + J$ is a subring of R stable by automorphisms of R.
(4) $C(R[x]) + J(R[x]) = C(R)[x] + N'[x]$ where $N' = J(R[x]) \cap R$ is a nil ideal of R. (Amitsur's result, see T.Y.Lam's book "first course" Theorem 5.10).
(5) CJ and CN rings are different notions for examples consider $R = k[[x]][[t;\sigma]]$ where σ is the k-endomorphism of $k[[x]]$ defined by $\sigma(x) = x^2$. The center of R is k and the Jacobson radical of R is the ideal generated by x and t. Hence R is CJ. But this ring is not CN since it is a noncommutative domain.

Let us mention some results related to CJ rings.

1. If R is CJ then R is Dedekind finite.
2. If R is CJ then $Nil(R) \subseteq J(R)$.
3. The subring $C + J$ is a CJ ring.
4. If $R[x]$ is a CJ ring, then R satisfies the Köthe conjecture.

THANK YOU !